Problem 4.32

(a) Find the eigenvalues and eigenspinors of S_y.

(b) If you measured S_y on a particle in the general state χ (Equation 4.139), what values might you get, and what is the probability of each? Check that the probabilities add up to 1. *Note:* a and b need not be real!

(c) If you measured S_y^2, what values might you get, and with what probabilities?

Solution

Part (a)

The matrix S_y is in Equation 4.147 on page 168.

$$\mathsf{S}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{bmatrix}$$

Start by finding its eigenvalues.

$$\det(\mathsf{S}_y - \lambda\mathsf{I}) = 0$$

$$\begin{vmatrix} -\lambda & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & -\lambda \end{vmatrix} = 0$$

$$(-\lambda)^2 - \left(-\frac{i\hbar}{2}\right)\left(\frac{i\hbar}{2}\right) = 0$$

$$\lambda^2 - \frac{\hbar^2}{4} = 0$$

$$\left(\lambda + \frac{\hbar}{2}\right)\left(\lambda - \frac{\hbar}{2}\right) = 0$$

As a result, the eigenvalues (the possible measurements of S_y) are

$$\lambda_- = -\frac{\hbar}{2} \quad \text{and} \quad \lambda_+ = \frac{\hbar}{2}.$$

Now find the eigenvectors (or rather eigenspinors in this context) associated with these eigenvalues.

$$(\mathsf{S}_y - \lambda_-\mathsf{I})\chi_-^{(y)} = 0 \qquad\qquad (\mathsf{S}_y - \lambda_+\mathsf{I})\chi_+^{(y)} = 0$$

$$\begin{bmatrix} \frac{\hbar}{2} & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & \frac{\hbar}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} -\frac{\hbar}{2} & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & -\frac{\hbar}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Write the system of equations implied from each matrix equation.

$$\left.\begin{aligned} \frac{\hbar}{2}x_1 - \frac{i\hbar}{2}x_2 = 0 \\ \frac{i\hbar}{2}x_1 + \frac{\hbar}{2}x_2 = 0 \end{aligned}\right\} \qquad \left.\begin{aligned} -\frac{\hbar}{2}x_1 - \frac{i\hbar}{2}x_2 = 0 \\ \frac{i\hbar}{2}x_1 - \frac{\hbar}{2}x_2 = 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} x_1 - ix_2 = 0 \\ ix_1 + x_2 = 0 \end{aligned}\right\} \qquad \left.\begin{aligned} -x_1 - ix_2 = 0 \\ ix_1 - x_2 = 0 \end{aligned}\right\}$$

$$x_2 = -ix_1 \qquad x_2 = ix_1$$

$$\chi_-^{(y)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -ix_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \chi_+^{(y)} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ ix_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Choose x_1 so that each eigenvector is normalized.

$$|x_1|^2 + |-ix_1|^2 = 1 \qquad |x_1|^2 + |ix_1|^2 = 1$$

$$x_1^2 + x_1^2 = 1 \qquad x_1^2 + x_1^2 = 1$$

$$2x_1^2 = 1 \qquad 2x_1^2 = 1$$

$$x_1^2 = \frac{1}{2} \qquad x_1^2 = \frac{1}{2}$$

$$x_1 = \frac{1}{\sqrt{2}} \qquad x_1 = \frac{1}{\sqrt{2}}$$

Therefore, the eigenvalues and associated normalized eigenspinors are

$$\begin{cases} \lambda_- = -\dfrac{\hbar}{2} \\ \chi_-^{(y)} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \end{cases} \quad \text{and} \quad \begin{cases} \lambda_+ = +\dfrac{\hbar}{2} \\ \chi_+^{(y)} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \end{cases}.$$

Part (b)

If you measure S_y on a particle with spin 1/2 in the general state,

$$\chi = \begin{bmatrix} a \\ b \end{bmatrix},$$

where $|a|^2 + |b|^2 = 1$, then you can get two results: $-\hbar/2$ and $+\hbar/2$.

The probability of measuring $-\hbar/2$ is the modulus squared of the component of $|\chi\rangle$ along $|\chi_-^{(y)}\rangle$.

$$\begin{aligned}
P\left(-\frac{\hbar}{2}\right) &= \left|c_-^{(y)}\right|^2 \\
&= \left|\langle\chi_-^{(y)} \mid \chi\rangle\right|^2 \\
&= \left|\chi_-^{(y)\dagger}\chi\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}^\dagger \begin{bmatrix}a\\b\end{bmatrix}\right|^2 \\
&= \frac{1}{2}\left|\begin{bmatrix}1 & i\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}\right|^2 \\
&= \frac{1}{2}|a+ib|^2 \\
&= \frac{1}{2}(a+ib)(a+ib)^* \\
&= \frac{1}{2}(a+ib)(a^*-ib^*) \\
&= \frac{1}{2}[aa^*+bb^*-i(ab^*-ba^*)] \\
&= \frac{1}{2}(|a|^2+|b|^2)+\frac{ab^*-(ab^*)^*}{2i} \\
&= \frac{1}{2}(|a|^2+|b|^2)+\operatorname{Im}(ab^*)
\end{aligned}$$

The probability of measuring $+\hbar/2$ is the modulus squared of the component of $|\chi\rangle$ along $|\chi_+^{(y)}\rangle$.

$$\begin{aligned}
P\left(+\frac{\hbar}{2}\right) &= \left|c_+^{(y)}\right|^2 \\
&= \left|\langle\chi_+^{(y)} \mid \chi\rangle\right|^2 \\
&= \left|\chi_+^{(y)\dagger}\chi\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}^\dagger \begin{bmatrix}a\\b\end{bmatrix}\right|^2
\end{aligned}$$

Simplify the result.

$$\begin{aligned} P\left(+\frac{\hbar}{2}\right) &= \frac{1}{2}\left|\begin{bmatrix}1 & -i\end{bmatrix}\begin{bmatrix}a \\ b\end{bmatrix}\right|^2 \\ &= \frac{1}{2}|a - ib|^2 \\ &= \frac{1}{2}(a - ib)(a - ib)^* \\ &= \frac{1}{2}(a - ib)(a^* + ib^*) \\ &= \frac{1}{2}[aa^* + bb^* + i(ab^* - ba^*)] \\ &= \frac{1}{2}(|a|^2 + |b|^2) - \frac{ab^* - (ab^*)^*}{2i} \\ &= \frac{1}{2}(|a|^2 + |b|^2) - \operatorname{Im}(ab^*) \end{aligned}$$

Therefore,

$$\begin{aligned} P\left(-\frac{\hbar}{2}\right) + P\left(+\frac{\hbar}{2}\right) &= \left[\frac{1}{2}(|a|^2 + |b|^2) + \operatorname{Im}(ab^*)\right] + \left[\frac{1}{2}(|a|^2 + |b|^2) - \operatorname{Im}(ab^*)\right] \\ &= |a|^2 + |b|^2 \\ &= 1. \end{aligned}$$

Part (c)

Compute the matrix S_y^2.

$$\mathsf{S}_y^2 = \mathsf{S}_y\mathsf{S}_y = \frac{\hbar}{2}\begin{bmatrix}0 & -i \\ i & 0\end{bmatrix}\frac{\hbar}{2}\begin{bmatrix}0 & -i \\ i & 0\end{bmatrix} = \frac{\hbar^2}{4}\begin{bmatrix}0 & -i \\ i & 0\end{bmatrix}\begin{bmatrix}0 & -i \\ i & 0\end{bmatrix} = \frac{\hbar^2}{4}\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{\hbar^2}{4} & 0 \\ 0 & \frac{\hbar^2}{4}\end{bmatrix}$$

Compute its eigenvalues.

$$\det(\mathsf{S}_y^2 - \lambda\mathsf{I}) = 0$$

$$\begin{vmatrix}\frac{\hbar^2}{4} - \lambda & 0 \\ 0 & \frac{\hbar^2}{4} - \lambda\end{vmatrix} = 0$$

$$\left(\frac{\hbar^2}{4} - \lambda\right)^2 = 0$$

$$\lambda = \frac{\hbar^2}{4}$$

Therefore, $+\hbar^2/4$ is the only possible value if S_y^2 is measured. The probability it occurs is 1.